Duration : 144 minutes

ÉCOLE POLYTECHNIQUE
FÉDÉRALE DE LAUSANNE

Linear Algebra Exam

Common part

Fall 2015

Answers

For the **multiple choice** questions, we give

$+3$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 if your answer is incorrect.

For the **true/false** questions, we give

$+1$ points if your answer is correct,

0 points if you give no answer or more than one,

-1 points if your answer is incorrect.

The notation and terminology of this exam are those used in the exercise sheets and the lectures of the course Linear Algebra given during the Fall semester 2015.

Notation

– For a matrix A, a_{ij} denotes the entry of A in row i and column j.
– For a vector $\boldsymbol{x}$, x_i denotes the i-the coordinate of $\boldsymbol{x}$.
– I_m denotes the $m \times m$ identity matrix.
– $\mathbb{P}_n$ is the vector space of polynomials of degree less than or equal to n.
– The inner product of vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ is defined as $\boldsymbol{x} \cdot \boldsymbol{y} = \boldsymbol{x}^T \boldsymbol{y}$.
– The length of a vector $\boldsymbol{x} \in \mathbb{R}^n$ is defined as $\|\boldsymbol{x}\| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$.

First part: Multiple choice questions

For each question, mark the box corresponding to the correct answer. Each question has **exactly one** correct answer.

Question 1 : Let A be a 5×6 matrix and let $\boldsymbol{b} \in \mathbb{R}^5$ be a vector. The set of solutions of the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$ cannot be equal to

■ a nonempty finite set
□ a subspace of $\mathbb{R}^6$ of dimension 1
□ a subspace of $\mathbb{R}^6$ of dimension 2
□ the empty set

Question 2 : Consider a set $\mathcal{S}$ of polynomials $p_1, \ldots, p_5$, such that the degree of p_k equals k for $k = 1, \ldots, 5$. Then

□ one can obtain a basis of $\mathbb{P}_5$ by adding the polynomial $p(t) = t^5 - t$ to the set $\mathcal{S}$
□ one can extract a basis of $\mathbb{P}_5$ from the set $\mathcal{S}$
□ the set $\mathcal{S}$ forms a basis of $\mathbb{P}_5$
■ one can obtain a basis of $\mathbb{P}_5$ by adding the polynomial $p(t) = 5$ to the set $\mathcal{S}$

Question 3 : Let A and B be two invertible matrices of the same size. If the matrices C and D are defined by $C = AB$ and $D = A^T + B^T$, then

□ D is always invertible, but C is not necessarily invertible
□ C is not necessarily invertible, and D is not necessarily invertible
□ C and D are always invertible
■ C is always invertible, but D is not necessarily invertible

Question 4 : Let

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & -1 & 2 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} 2 \\ 2h-4 \\ -3-h \end{pmatrix},$$

where h is a real parameter. Then the matrix equation $A\boldsymbol{x} = \boldsymbol{b}$

■ has infinitely many solutions for $h = 3$
□ has infinitely many solutions for $h = -3$
□ has finitely many solutions for every $h \in \mathbb{R}$
□ has infinitely many solutions for every $h \neq \pm3$

Question 5 : Let A be an $n \times n$ matrix. Three of the following four statements are equivalent to each other. Which statement is not one of those three?

- ☐ The set of columns of A forms an orthonormal basis of $\mathbb{R}^n$
- ■ $\det A = 1$
- ☐ The set of rows of A forms an orthonormal basis of $\mathbb{R}^n$
- ☐ $AA^T = I_n$

Question 6 : Let B be an $m \times n$ matrix such that $BB^T = I_m$. Then

- ☐ the columns of B form an orthonormal set
- ■ the rows of B form an orthonormal set
- ☐ $B^T B = I_n$
- ☐ B is invertible

Question 7 : Let $A = \begin{pmatrix} 7 & -24 \\ -24 & -7 \end{pmatrix}$. There is an orthogonal diagonalization $A = PDP^T$, where D is a diagonal matrix, and where

- ☐ $P = \begin{pmatrix} 4/5 & 3/5 \\ 3/5 & -4/5 \end{pmatrix}$
- ☐ $P = \begin{pmatrix} 5/13 & 12/13 \\ 12/13 & -5/13 \end{pmatrix}$
- ■ $P = \begin{pmatrix} 3/5 & 4/5 \\ 4/5 & -3/5 \end{pmatrix}$
- ☐ $P = \begin{pmatrix} 12/13 & 5/13 \\ 5/13 & -12/13 \end{pmatrix}$

Question 8 : There exists a polynomial $p(t) = a_0 + a_1 t + a_2 t^2$ with real coefficients a_0, a_1, a_2 such that

$$p(-1) = 1, \qquad p(0) = 1, \qquad p(1) = 3, \qquad p(2) = b,$$

- ■ for one single real value of b
- ☐ for no real value of b
- ☐ for every real value of b
- ☐ for a finite set of at least two real values of b

Question 9 : The area of the triangle determined by the three points

$$(0,0), \quad (-1,5), \quad (3,-1)$$

equals

- ■ 7
- ☐ 8
- ☐ 16
- ☐ 14

Question 10 : Let

$$\boldsymbol{x} = \begin{pmatrix} -1 \\ 3 \\ 3 \\ 1 \end{pmatrix}, \quad \boldsymbol{u} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \quad \text{et} \quad \boldsymbol{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}.$$

Then the orthogonal projection of $\boldsymbol{x}$ on the subspace Span $\{\boldsymbol{u}, \boldsymbol{v}\}$ is:

☐ $\begin{pmatrix} -1 \\ 3 \\ 3 \\ 1 \end{pmatrix}$ ■ $\begin{pmatrix} 1 \\ 2 \\ 3 \\ 0 \end{pmatrix}$ ☐ $\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ ☐ $\begin{pmatrix} 2 \\ 3 \\ 3 \\ 1 \end{pmatrix}$

Question 11 : The value of the parameter $b \in \mathbb{R}$, for which the vector $\boldsymbol{w} = \begin{pmatrix} 4 \\ b \\ 0 \end{pmatrix}$ lies in the plane in $\mathbb{R}^3$ spanned by the vectors $\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ and $\boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$, is:

☐ $b = 2$
☐ $b = -1$
■ $b = -5$
☐ $b = -2$

Question 12 : Let A be an $m \times n$ matrix, $\boldsymbol{b}$ a vector in $\mathbb{R}^m$, and $\widehat{\boldsymbol{b}}$ the orthogonal projection of $\boldsymbol{b}$ on Col(A). Then

☐ the least-squares solution of $A\boldsymbol{x} = \boldsymbol{b}$ is $A^{-1}\widehat{\boldsymbol{b}}$
☐ the matrix A^TA is invertible
■ every solution of $A^TA\boldsymbol{x} = A^T\boldsymbol{b}$ is a least-squares solution of $A\boldsymbol{x} = \boldsymbol{b}$
☐ the equation $A\boldsymbol{x} = \widehat{\boldsymbol{b}}$ has a unique solution

Question 13 : The set of symmetric 2×2 matrices forms a vector space with the usual rules for adding matrices and for multiplying a matrix by a scalar. Let

$$A = \begin{pmatrix} 5 & -3 \\ -3 & 1 \end{pmatrix} \quad \text{be a matrix and let} \quad \mathcal{B} = \left\{ \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

be a basis of this vector space. If we represent A in the basis $\mathcal{B}$, then the coefficient of the third vector of $\mathcal{B}$ is

■ 9
☐ -3
☐ 2
☐ 1

Question 14 : The determinant of the matrix

$$\begin{pmatrix} 2 & 3 & 0 & 1 \\ 0 & 2 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 5 & 4 & 3 & -2 \end{pmatrix}$$

equals

- ☐ 24
- ☐ 12
- ☐ -12
- ■ -24

Question 15 : Let A and B be $n \times n$ matrices, with B invertible. Suppose that λ is an eigenvalue of A, and also an eigenvalue of B.
Among the following statements:

(a) λ is an eigenvalue of the matrix $A + B$,

(b) λ is an eigenvalue of the matrix AB,

(c) λ is an eigenvalue of the matrix BAB^{-1},

(d) λ^2 is an eigenvalue of the matrix BA,

which are always true?

- ■ only (c)
- ☐ (a), (c), and (d)
- ☐ only (d)
- ☐ (a) and (b)

Question 16 : Let

$$A = \begin{pmatrix} 1 & -3 & 2 \\ 5 & 0 & 1 \\ -2 & 11 & 0 \end{pmatrix}.$$

Compute the LU factorization of A (using only the elementary row operation that adds a multiple of one row to another row below it). Then the entry ℓ_{31} of L equals

- ☐ 2
- ☐ 3
- ☐ -1
- ■ -2

Question 17 : Let

$$A = \begin{pmatrix} 0 & 0 & 5 \\ 0 & 3 & -3 \\ 2 & 6 & -4 \end{pmatrix}.$$

If $B = A^{-1}$, then the entry b_{11} of B equals

- ☐ -1
- ■ $-\frac{1}{5}$
- ☐ $\frac{1}{5}$
- ☐ $\frac{1}{3}$

Question 18 : Consider the matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix}$ and the vector $\boldsymbol{b} = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \end{pmatrix}$.

Then the least-squares solution $\widehat{\boldsymbol{x}} = \begin{pmatrix} \widehat{x}_1 \\ \widehat{x}_2 \end{pmatrix}$ of the equation $A\boldsymbol{x} = \boldsymbol{b}$ satisfies

- ■ $\widehat{x}_2 = 3$
- ☐ $\widehat{x}_2 = -3$
- ☐ $\widehat{x}_2 = 4$
- ☐ $\widehat{x}_2 = -4$

Question 19 : Consider the transformation $T : \mathbb{R}^3 \to \mathbb{P}_2$ defined by the formula

$$T\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) = (c - b) + (a - 2b + c)t + (a - b)t^2\,.$$

Then

- ☐ T is linear and its rank equals 1
- ■ T is linear and its rank equals 2
- ☐ T is linear and one-to-one
- ☐ T is not linear

Question 20 : Let h be a real parameter. The vectors

$$\begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \\ h \\ 1 \end{pmatrix}$$

form a basis of $\mathbb{R}^4$

- ☐ only if $h = 0$
- ■ only if $h \neq \frac{1}{2}$
- ☐ only if $h = \frac{1}{2}$
- ☐ only if $h \neq 0$

Question 21 : The system of linear equations

$$\begin{cases} -x_1 \phantom{{}-x_2} - 2x_3 \phantom{{}+x_4} = 0 \\ x_2 + x_3 + x_4 = 0 \\ x_1 - x_2 + x_3 - x_4 = 0 \\ -2x_2 - 2x_3 - 3x_4 = 0 \end{cases}$$

☐ does not have a free variable

☐ has exactly two free variables

■ has exactly one free variable

☐ has exactly three free variables

Question 22 : Let

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}.$$

Then

☐ the only eigenvalue of A is 2

☐ the eigenvalues of A are 0, 2, and -2

☐ the eigenvalues of A are 1, -1, 2, and -2

■ the eigenvalues of A are 2 and -2

Question 23 : Let

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 7 & -2 \end{pmatrix}.$$

Then

☐ A has three distinct eigenvalues

☐ A is orthogonally diagonalizable

☐ A is diagonalizable but not orthogonally diagonalizable

■ A is not diagonalizable

Question 24 : Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by
$T\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = \begin{pmatrix} 4x_3 \\ 3x_1 + 5x_2 - 2x_3 \\ x_1 + x_2 + 4x_3 \end{pmatrix}$. Let $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$.
Let M be the matrix that represents T in the basis $\mathcal{B}$; in other words, we have $\left[T(\boldsymbol{x})\right]_{\mathcal{B}} = M\left[\boldsymbol{x}\right]_{\mathcal{B}}$ for every $\boldsymbol{x} \in \mathbb{R}^3$. Then

☐ $M = \begin{pmatrix} 4 & 6 & 6 \\ 0 & 8 & 2 \\ 0 & 3 & 1 \end{pmatrix}$

☐ $M = \begin{pmatrix} 6 & 0 & -2 \\ 2 & 6 & -8 \\ 1 & 2 & -3 \end{pmatrix}$

■ $M = \begin{pmatrix} 6 & 2 & 1 \\ 0 & 6 & 2 \\ -2 & -8 & -3 \end{pmatrix}$

☐ $M = \begin{pmatrix} 4 & 0 & 0 \\ 6 & 8 & 3 \\ 6 & 2 & 1 \end{pmatrix}$

Second part: true/false questions

For each question, mark (without erasures) TRUE if the statement is **always true** and FALSE if it is **not always true** (i.e., it is sometimes false).

Question 25 : Let $p \in \mathbb{P}_n$ be a polynomial and let p' be the derivative of p.
The set $\{p \in \mathbb{P}_n : p'(-1) \neq 0\}$ is a subspace of $\mathbb{P}_n$.

☐ TRUE ■ FALSE

Question 26 : Let A be an $n \times n$ matrix and λ a non-zero real number. If the homogeneous system $A\boldsymbol{x} = \boldsymbol{0}$ has exactly one solution, then the set of columns of $(\lambda A)^T$ spans $\mathbb{R}^n$.

■ TRUE ☐ FALSE

Question 27 : If A and B are row equivalent matrices (in other words, B can be obtained from A using elementary row operations), then $\mathrm{Col}(A^T) = \mathrm{Col}(B^T)$.

■ TRUE ☐ FALSE

Question 28 : Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation that is one-to-one. Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p\}$ be a linearly independent set of vectors in $\mathbb{R}^n$. Then $\{T(\boldsymbol{v}_1), T(\boldsymbol{v}_2), \ldots, T(\boldsymbol{v}_p)\}$ is a linearly independent set of vectors in $\mathbb{R}^m$.

■ TRUE ☐ FALSE

Question 29 : Let $T\colon \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a linear transformation, and let A be the matrix that represents T in the standard basis. If T is onto, then the null space of A has dimension 1.

■ TRUE ☐ FALSE

Question 30 : Let A be a 7×3 matrix, whose first two columns are linearly independent, and whose third column equals its first. Then the matrix A^TA has size 3×3, is symmetric, and has rank 2.

■ TRUE ☐ FALSE

Question 31 : If A is a 4×4 matrix of rank 1, and $\lambda = 0$ is an eigenvalue of A with algebraic multiplicity 3, then A is diagonalizable. (The algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial of the matrix.)

■ TRUE ☐ FALSE

Question 32 : Let A be an $m \times n$ matrix. If $\boldsymbol{x} \in \mathrm{Col}(A)$ satisfies $A^T\boldsymbol{x} = \boldsymbol{0}$, then $\|\boldsymbol{x}\| = 0$.

■ TRUE ☐ FALSE